

Can brains generate random numbers?

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Abstract

Motivated by EEG recordings of normal brain activity, we construct arbitrarily large McCulloch-Pitts neural networks that, without any external input, make every subset of their neurons fire in some iteration (and therefore in infinitely many iterations).

1 Introduction

Epilepsy is a group of neurologic conditions, the common and fundamental characteristic of which is recurrent, unprovoked epileptic seizures. These seizures are transient changes in attention or behavior, often accompanied by convulsions; they result from excessive, abnormal firing patterns of neurons that are located predominantly in the cerebral cortex (the convoluted outer layer of gray matter that covers each cerebral hemisphere). Such abnormal paroxysmal activity is usually intermittent and self-limited. [4, p.2]. The World Health Organization reports [26] that there are over 50 million epilepsy sufferers in the world today, 85% of whom live in developing countries.

In attempts to study epilepsy, selected patients are monitored continuously for days at a time. During these periods, EEG (electroencephalogram) or ECoG (electrocorticogram) recordings are made. EEG recordings come from placing multiple electrodes on the scalp of the patient. ECoG recordings produce far more accurate data, but they require invasive surgery to place a grid of electrodes directly on the cortex.

Neurologists specialized in epilepsy are trained to read EEG/ECoG recordings, so that mere visual inspection allows them to tell with a reasonable degree of accuracy when a seizure might have occurred. There are a number of different types of seizures; two major categories are *partial seizures* (originating in a small group of neurons, called a seizure focus, and spreading to other brain regions) and *generalized seizures* (showing simultaneous disruption of normal brain activity in both cerebral hemispheres from the onset) [25]. The classification [3] developed by the International League Against Epilepsy in

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1981 divides each of these two categories into several subcategories, some of which are divided into subsubcategories or even further and these different types of seizures have different EEG manifestations [5, 20]. One frequent occurrence is a transition from an irregular, disorderly EEG before the seizure (the pre-ictal state) to more organized sustained rhythm of spikes or sharp waves during the seizure (the ictal state) [2, Chapter 2]. (Some researchers refer to the pre-ictal EEG informally as ‘chaotic’ or ‘random’ in contrast with the rigorous definitions of these modifiers in mathematics, where — roughly speaking — ‘chaotic’ means ‘highly dependent on initial conditions’ and ‘random’ means ‘unpredictable’.)

In July 2009, in a seminar held at Concordia, Nithum Thain asked whether some initial configuration could cause Conway’s Game of Life [7] to evolve in a way resembling a partial seizure, proceeding from an erratic flutter of apparently unpredictable patterns to sustained rhythmic changes that would begin in a small part of the grid and gradually spread, synchronized, over a larger area before subsiding to give way to the initial erratic mode. In the discussion that followed, a variation emerged: Could a McCulloch-Pitts neural network behave like this?

To define these networks, we need first the notion of a *linear threshold function*. This is a function $f : \mathbf{R}^n \rightarrow \{0, 1\}$ such that, for some real numbers w_1, w_2, \dots, w_n (mnemonic for “weights”) and θ (mnemonic for “threshold”),

$$f(x_1, x_2, \dots, x_n) = \begin{cases} 1 & \text{if } \sum_{j=1}^n w_j x_j \geq \theta, \\ 0 & \text{otherwise.} \end{cases}$$

The function is thought of as a neuron with zero-one signals x_1, x_2, \dots, x_n received at its synapses from the axons of other neurons; positive weights correspond to excitatory synapses and negative weights correspond to inhibitory synapses; $f(x_1, x_2, \dots, x_n) = 1$ means that the neuron, given signals x_1, x_2, \dots, x_n at time t , will fire (send the signal ‘one’) along its axon after a synaptic delay at time $t + 1$.

A *McCulloch-Pitts neural network* [17], with nonnegative integer parameters p and n such that $p < n$, is a collection of linear threshold functions

$$f_i : \{0, 1\}^n \rightarrow \{0, 1\} \quad (i = p + 1, p + 2, \dots, n).$$

Given any zero-one vector $s_{p+1}, s_{p+2}, \dots, s_n$ (the initial state of the network) and any sequence $\xi_1, \xi_2, \dots, \xi_p$ of p functions

$$\xi_r : \mathbf{N} \rightarrow \{0, 1\} \quad (r = 1, 2, \dots, p),$$

it computes a sequence $x_{p+1}, x_{p+2}, \dots, x_n$ of $n - p$ functions

$$x_i : \mathbf{N} \rightarrow \{0, 1\} \quad (i = p + 1, p + 2, \dots, n.)$$

This is done by setting $x_i(0) = s_i$ and, for all nonnegative integers t ,

$$x_i(t + 1) = f_i(\xi_1(t), \dots, \xi_p(t), x_{p+1}(t), \dots, x_n(t)).$$

We think of variable t as marking discrete time; each of the $n - p$ neurons $p + 1, p + 2, \dots, n$ may receive its signals from any of the n neurons; it receives a signal from neuron j if and only if its w_j is nonzero. The bits $\xi_r(t)$ with $r = 1, 2, \dots, p$ and the bits $x_i(t)$ with $i = p + 1, p + 2, \dots, n$ tell us which neurons are firing at time t ; firing or not firing of a neuron at time $t + 1$ depends on the signals arriving to it at time t . Neurons $1, 2, \dots, p$ receive no signals; they have no axons synapsing upon them; McCulloch and Pitts call them ‘the peripheral afferents’ of the network. We will restrict ourselves to McCulloch-Pitts neural networks with no such peripheral afferents; to put it differently, we will set $p = 0$. The entire network is then a function $\Phi : \{0, 1\}^n \rightarrow \{0, 1\}^n$ defined by

$$\Phi(x) = (f_1(x), f_2(x), \dots, f_n(x))$$

and $(x_1(t), x_2(t), \dots, x_n(t)) = \Phi^t(s_1, s_2, \dots, s_n)$ for all nonnegative integers t .

The McCulloch-Pitts concept of an artificial neuron has been generalized to allow firing at intensities on a continuous scale rather than in the all-or-none way: instead of being a linear threshold function, each $f_i : \mathbf{R}^n \rightarrow \mathbf{R}$ is now defined by

$$f_i(x_1, x_2, \dots, x_n) = \varphi_i(\sum_{j=1}^n w_{ij}x_j)$$

with some ‘transfer function’ $\varphi_i : \mathbf{R} \rightarrow \mathbf{R}$; the McCulloch-Pitts original special case has $\varphi_i(s) = H(s - \theta_i)$, where H is the *Heaviside step function* defined by

$$H(d) = \begin{cases} 1 & \text{when } d \geq 0, \\ 0 & \text{when } d < 0. \end{cases}$$

All such mathematical abstractions of biological neurons are only crude approximations of their actual behaviour. More credible models are *spiking neurons*, which were anticipated by Lapicque [12, 1] long before the mechanisms of the generation of neuronal action potentials were known; later on, a

different model was proposed by Hodgkin and Huxley [9] and subsequently elaborated by many other researchers [24, 21, 11, 16, 8].

We restrict ourselves to the McCulloch-Pitts model (with no peripheral afferents) in all its simplicity: it played a seminal role in the development of artificial neural networks [22, 23] and even today it is routinely referenced in medical literature ([10, 19, 27] are just three of the more recent citations). We do not pretend that our findings have any biological significance, but we hope that they may serve as a template for generalizations to more realistic models of the brain, such as networks of spiking neurons.

2 The question

Could a McCulloch-Pitts neural network simulate a partial seizure? We have replaced this question by its easier variation: Could such a network simulate the pre-ictal state of a brain? To put it differently, are there irregular, disorderly, apparently unpredictable McCulloch-Pitts networks? An essential prerequisite of every such network $\Phi : \{0, 1\}^n \rightarrow \{0, 1\}^n$ is that, starting from any state s in its domain $\{0, 1\}^n$, it eventually produce every state in this domain as an element of the trajectory $s, \Phi(s), \Phi^2(s), \dots$. This means that the *period of* Φ , defined as the smallest t such that $\Phi^{t+1}(s) = s$ for some s in its domain, equals the size of the domain. This observation leads us to ask an even easier question: Are there McCulloch-Pitts networks $\Phi : \{0, 1\}^n \rightarrow \{0, 1\}^n$ with period 2^n ? (The experimental study [18] of trajectories in randomly generated McCulloch-Pitts networks is to some extent related to this question; lengths of state cycles in Boolean networks, cellular automata, and other finite dynamical systems are often interpreted as a measure of their computational power [13].) The result reported here is that the answer is affirmative for every positive integer n . (In addition, we have found a number of properties that threshold functions, and tuples of threshold functions, must possess in order to define such networks. These will be the subject of a subsequent paper.)

3 The answer

Given a positive integer n , we define a mapping $\Phi_n : \{0, 1\}^n \rightarrow \{0, 1\}^n$ by $\Phi_n(x_1, x_2, \dots, x_n) = (y_1, y_2, \dots, y_n)$ where, with m the largest subscript such

that (x_1, x_2, \dots, x_m) is an alternating vector, $(0, 1, 0, 1, \dots)$ or $(1, 0, 1, 0, \dots)$,

$$y_i = \begin{cases} \bar{x}_m & \text{when } 1 \leq i \leq m, \\ x_i & \text{when } m < i \leq n. \end{cases} \quad (1)$$

(Here, as usual, \bar{x} with $x \in \{0, 1\}$ denotes $1 - x$.) For instance,

$$\begin{aligned} \Phi_4(0, 0, 0, 0) &= (1, 0, 0, 0), & \Phi_4(0, 0, 0, 1) &= (1, 0, 0, 1), \\ \Phi_4(0, 0, 1, 0) &= (1, 0, 1, 0), & \Phi_4(0, 0, 1, 1) &= (1, 0, 1, 1), \\ \Phi_4(0, 1, 0, 0) &= (1, 1, 1, 0), & \Phi_4(0, 1, 0, 1) &= (0, 0, 0, 0), \\ \Phi_4(0, 1, 1, 0) &= (0, 0, 1, 0), & \Phi_4(0, 1, 1, 1) &= (0, 0, 1, 1), \\ \Phi_4(1, 0, 0, 0) &= (1, 1, 0, 0), & \Phi_4(1, 0, 0, 1) &= (1, 1, 0, 1), \\ \Phi_4(1, 0, 1, 0) &= (1, 1, 1, 1), & \Phi_4(1, 0, 1, 1) &= (0, 0, 0, 1), \\ \Phi_4(1, 1, 0, 0) &= (0, 1, 0, 0), & \Phi_4(1, 1, 0, 1) &= (0, 1, 0, 1), \\ \Phi_4(1, 1, 1, 0) &= (0, 1, 1, 0), & \Phi_4(1, 1, 1, 1) &= (0, 1, 1, 1) \end{aligned}$$

Note that the definition of Φ_n implies that Φ_n is antisymmetric in the sense of

$$\Phi_n(\bar{x}) = \overline{\Phi_n(x)} \text{ for all } x \text{ in } \{0, 1\}^n \quad (2)$$

and that

$$\Phi_1(0) = 1 \quad (3)$$

and that, when $n \geq 2$,

$$\Phi_n(x_1, x_2, \dots, x_{n-1}, 0) = \begin{cases} (1, 1, \dots, 1, 1) & \text{if } (x_1, x_2, \dots, x_{n-1}, 0) \text{ is} \\ \text{the alternating vector } (\dots, 1, 0, 1, 0), & \\ (\Phi_{n-1}(x_1, x_2, \dots, x_{n-1}), 0) & \text{otherwise.} \end{cases} \quad (4)$$

Theorem 1. *The period of Φ_n is 2^n .*

Proof. Straightforward induction on n , using properties (2), (3), (4), proves a finer statement: The 2^n vectors $\Phi_n^t(0, 0, \dots, 0)$ with $t = 0, 1, \dots, 2^n - 1$ are pairwise distinct and $\Phi_n^{2^n-1}(0, 0, \dots, 0)$ is the alternating vector $(\dots, 0, 1, 0, 1)$. \square

Theorem 2. *For every positive integer n there are linear threshold functions*

$$f_{n,i} : \{0, 1\}^n \rightarrow \{0, 1\} \quad (i = 1, 2, \dots, n)$$

such that

$$\Phi_n(x) = (f_{n,1}(x), f_{n,2}(x), \dots, f_{n,n}(x)) \text{ for all } x \text{ in } \{0, 1\}^n. \quad (5)$$

Proof. For every positive integer n and for all $i = 1, 2, \dots, n$, we will construct weights $w_{n,i,j}$ ($j = 1, 2, \dots, n$) and threshold values $\theta_{n,i}$. Then we will define

$$f_{n,i}(x_1, \dots, x_n) = H\left(\sum_{j=1}^n w_{n,i,j}x_j - \theta_{n,i}\right)$$

and prove that (5) is satisfied.

Our construction of $w_{n,i,j}$ and $\theta_{n,i}$ is recursive. To begin, we set

$$w_{1,1,1} = -1, \theta_{1,1} = 0;$$

for all integers n greater than 1, we set

$$w_{n,n,j} = \begin{cases} 1 & \text{if } j \not\equiv n \pmod{2}, \\ -1 & \text{if } j \equiv n \pmod{2} \text{ and } j < n, \\ n-2 & \text{if } j = n, \end{cases}$$

$$\theta_{n,n} = \lfloor n/2 \rfloor,$$

$$w_{n,n-1,j} = \begin{cases} w_{n-1,n-1,j} & \text{if } j \leq n-2, \\ w_{n-1,n-1,j} + 1 & \text{if } j = n-1, \\ -1 & \text{if } j = n, \end{cases}$$

$$\theta_{n,n-1} = \theta_{n-1,n-1},$$

and, when $i = 1, 2, \dots, n-2$,

$$w_{n,i,j} = \begin{cases} w_{n-1,i,j} + w_{n-2,i,j} & \text{if } j \leq n-2, \\ w_{n-1,i,j} & \text{if } j = n-1, \\ -1 & \text{if } j = n, \end{cases}$$

$$\theta_{n,i} = \theta_{n-1,i} + \theta_{n-2,i} - 1.$$

Since the sequence $\Phi_1, \Phi_2, \Phi_3, \dots$ is completely determined by its properties (2), (3), (4), proving that (5) is satisfied reduces to proving that

$$(i) \quad f_{n,i}(\bar{x}) = \overline{f_{n,i}(x)} \text{ for all } i = 1, \dots, n \text{ and all } x \text{ in } \{0, 1\}^n,$$

observing that $f_{1,1}(0) = 1$, and proving that

$$(ii) \quad \text{if } x \text{ in } \{0, 1\}^n \text{ is the alternating vector } (\dots, 1, 0, 1, 0), \\ \text{then } f_{n,i}(x) = 1 \text{ for all } i = 1, \dots, n,$$

- (iii) if $(x_1, \dots, x_{n-1}, 0)$ in $\{0, 1\}^n$ is not the alternating vector $(\dots, 1, 0, 1, 0)$, then $f_{n,n}(x_1, \dots, x_{n-1}, 0) = 0$,
- (iv) if $(x_1, \dots, x_{n-1}, 0)$ in $\{0, 1\}^n$ is not the alternating vector $(\dots, 1, 0, 1, 0)$, then $f_{n,i}(x_1, \dots, x_{n-1}, 0) = f_{n-1,i}(x_1, \dots, x_{n-1})$ for all $i = 1, \dots, n-1$.

Straightforward, if a little tedious, induction on n shows that

$$\sum_{j \not\equiv n \pmod 2} w_{n,i,j} = \theta_{n,i} \quad \text{for all } i = 1, \dots, n, \quad (6)$$

$$\sum_{j \equiv n \pmod 2} w_{n,i,j} = \theta_{n,i} - 1 \quad \text{for all } i = 1, \dots, n. \quad (7)$$

Summing up each pair of these equations, we conclude that

$$\sum_{j=1}^n w_{n,i,j} = 2\theta_{n,i} - 1 \text{ for all } i = 1, \dots, n,$$

which is easily seen to imply (i); equations (6) alone imply directly (ii); proposition (iii) follows from the definitions of $w_{n,n,j}$ and $\theta_{n,n}$.

In proving (iv), we will treat $i = n-1$ separately from $i \leq n-2$.

To prove that $f_{n,n-1}(x_1, x_2, \dots, x_{n-1}, 0) = f_{n-1,n-1}(x_1, x_2, \dots, x_{n-1})$ for all (x_1, \dots, x_{n-1}) in $\{0, 1\}^{n-1}$ other than the alternating vector $(\dots, 0, 1, 0, 1)$, recall that

$$\sum_{j=1}^{n-1} w_{n,n-1,j} x_j = \sum_{j=1}^{n-1} w_{n-1,n-1,j} x_j + x_{n-1} \text{ and } \theta_{n,n-1} = \theta_{n-1,n-1}.$$

It follows that $f_{n,n-1}(x_1, \dots, x_{n-1}, 0) \neq f_{n-1,n-1}(x_1, \dots, x_{n-1})$ if and only if $x_{n-1} = 1$ and $\sum_{j=1}^{n-1} w_{n-1,n-1,j} x_j + 1 = \theta_{n-1,n-1}$. Since $w_{n-1,n-1,n-1} = n-3$ and $\theta_{n-1,n-1} = \lfloor (n-1)/2 \rfloor$, this means $\sum_{j=1}^{n-2} w_{n-1,n-1,j} x_j = -\lfloor (n-3)/2 \rfloor$; since

$$w_{n-1,n-1,j} = \begin{cases} 1 & \text{if } j \not\equiv n-1 \pmod 2, \\ -1 & \text{if } j \equiv n-1 \pmod 2 \text{ and } j < n-1, \end{cases}$$

this means further that (x_1, \dots, x_{n-1}) is the alternating vector $(\dots, 0, 1, 0, 1)$.

To prove that we have $f_{n,i}(x_1, \dots, x_{n-1}, 0) = f_{n-1,i}(x_1, \dots, x_{n-1})$ for all $i = 1, \dots, n-2$ and for all (x_1, \dots, x_{n-1}) in $\{0, 1\}^{n-1}$ other than the alternating vector $(\dots, 0, 1, 0, 1)$, we shall use induction on n . In the induction step, we distinguish between two cases.

CASE 1: (x_1, \dots, x_{n-1}) is the alternating vector $(\dots, 1, 0, 1, 0)$.

In this case, we do not use the induction hypothesis. Equations (7) show that

$$\sum_{j=1}^{n-1} w_{n,i,j} x_j + w_{n,i,n} = \theta_{n,i} - 1 \quad \text{for all } i = 1, \dots, n;$$

since $w_{n,i,n} = -1$ for all $i = 1, \dots, n-1$, it follows that

$$\sum_{j=1}^{n-1} w_{n,i,j} x_j = \theta_{n,i} \quad \text{for all } i = 1, \dots, n-1,$$

and so $f_{n,i}(x_1, \dots, x_{n-1}, 0) = 1$ for all $i = 1, \dots, n-1$. By (ii) with $n-1$ in place of n , we have $f_{n-1,i}(x_1, \dots, x_{n-1}) = 1$ for all $i = 1, \dots, n-1$.

CASE 2: (x_1, \dots, x_{n-1}) is not the alternating vector $(\dots, 1, 0, 1, 0)$.

In this case, consider an arbitrary $(x_1, \dots, x_{n-1}, 0)$ in $\{0, 1\}^n$ other than the alternating vector $(\dots, 1, 0, 1, 0)$. The induction hypothesis guarantees (alone if $x_{n-1} = 0$ and combined with (i) if $x_{n-1} = 1$) that $f_{n-1,i}(x_1, \dots, x_{n-1}) = f_{n-2,i}(x_1, \dots, x_{n-2})$ for all $i = 1, \dots, n-2$.

If $i \leq n-2$ and $f_{n-1,i}(x_1, \dots, x_{n-1}) = 1$, then $f_{n-2,i}(x_1, \dots, x_{n-2}) = 1$, and so

$$\begin{aligned} \sum_{j=1}^{n-1} w_{n,i,j} x_j &= \sum_{j=1}^{n-1} w_{n-1,i,j} x_j + \sum_{j=1}^{n-2} w_{n-2,i,j} x_j \\ &\geq \theta_{n-1,i} + \theta_{n-2,i} > \theta_{n,i}, \end{aligned}$$

which implies $f_{n,i}(x_1, \dots, x_{n-1}, 0) = 1$.

If $i \leq n-2$ and $f_{n-1,i}(x_1, \dots, x_{n-1}) = 0$, then $f_{n-2,i}(x_1, \dots, x_{n-2}) = 0$, and so

$$\begin{aligned} \sum_{j=1}^{n-1} w_{n,i,j} x_j &= \sum_{j=1}^{n-1} w_{n-1,i,j} x_j + \sum_{j=1}^{n-2} w_{n-2,i,j} x_j \\ &\leq (\theta_{n-1,i} - 1) + (\theta_{n-2,i} - 1) < \theta_{n,i}, \end{aligned}$$

which implies $f_{n,i}(x_1, \dots, x_{n-1}, 0) = 0$. \square

Implicit in our proof of Theorem 1 is a simple way of transforming each trajectory

$$(0, 0, \dots, 0) \mapsto \Phi_n(0, 0, \dots, 0) \mapsto \dots \Phi_n^{N-1}(0, 0, \dots, 0) \quad (8)$$

with $N = 2^n$ into the trajectory

$$(0, 0, \dots, 0) \mapsto \Phi_{n+1}(0, 0, \dots, 0) \mapsto \dots \Phi_{n+1}^{2N-1}(0, 0, \dots, 0) : \quad (9)$$

First append 0 as the last bit to each point of the trajectory (8) and let T denote the resulting sequence of 2^n vectors $(x_1, \dots, x_n, 0)$ in $\{0, 1\}^{n+1}$; then flip every bit ($0 \leftrightarrow 1$) of every vector in T and let \overline{T} denote the resulting sequence of 2^n vectors $(x_1, \dots, x_n, 1)$ in $\{0, 1\}^{n+1}$; the trajectory (9) is the concatenation $T\overline{T}$. For instance, if $n = 3$, then (8) is

$$\begin{aligned} (0, 0, 0) \mapsto (1, 0, 0) \mapsto (1, 1, 0) \mapsto (0, 1, 0) \mapsto \\ (1, 1, 1) \mapsto (0, 1, 1) \mapsto (0, 0, 1) \mapsto (1, 0, 1), \end{aligned}$$

T is

$$\begin{aligned} (0, 0, 0, 0) \mapsto (1, 0, 0, 0) \mapsto (1, 1, 0, 0) \mapsto (0, 1, 0, 0) \mapsto \\ (1, 1, 1, 0) \mapsto (0, 1, 1, 0) \mapsto (0, 0, 1, 0) \mapsto (1, 0, 1, 0), \end{aligned}$$

\overline{T} is

$$\begin{aligned} (1, 1, 1, 1) \mapsto (0, 1, 1, 1) \mapsto (0, 0, 1, 1) \mapsto (1, 0, 1, 1) \mapsto \\ (0, 0, 0, 1) \mapsto (1, 0, 0, 1) \mapsto (1, 1, 0, 1) \mapsto (0, 1, 0, 1), \end{aligned}$$

and (9) is

$$\begin{aligned} (0, 0, 0, 0) \mapsto (1, 0, 0, 0) \mapsto (1, 1, 0, 0) \mapsto (0, 1, 0, 0) \mapsto \\ (1, 1, 1, 0) \mapsto (0, 1, 1, 0) \mapsto (0, 0, 1, 0) \mapsto (1, 0, 1, 0) \mapsto \\ (1, 1, 1, 1) \mapsto (0, 1, 1, 1) \mapsto (0, 0, 1, 1) \mapsto (1, 0, 1, 1) \mapsto \\ (0, 0, 0, 1) \mapsto (1, 0, 0, 1) \mapsto (1, 1, 0, 1) \mapsto (0, 1, 0, 1). \end{aligned}$$

It may be interesting to note that our Φ_n can be specified in yet another way. Every one-to-one mapping $r : \{0, 1\}^n \rightarrow \{0, 1, \dots, 2^n - 1\}$ generates a mapping $\Phi : \{0, 1\}^n \rightarrow \{0, 1\}^n$ through the formula

$$\Phi(x) = r^{-1}(r(x) + 1 \bmod 2^n).$$

We have, for $s = r^{-1}(0)$ and for all $t = 0, 1, \dots, 2^n - 1$,

$$x = \Phi^t(s) \Leftrightarrow t = r(x)$$

(this can be checked by straightforward induction on t); it follows that Φ has period 2^n . Our Φ_n is generated by the mapping $r_n : \{0, 1\}^n \rightarrow \{0, 1, \dots, 2^n - 1\}$ defined by $r_n(x_1, x_2, \dots, x_n) = \sum_{j=1}^n c_j 2^{j-1}$ with

$$c_j = \begin{cases} |x_j - x_{j+1}| & \text{when } 1 \leq j < n, \\ x_n & \text{when } j = n. \end{cases}$$

For instance,

$$\begin{aligned} r_4(0,0,0,0) &= 0, & r_4(0,0,0,1) &= 12, & r_4(0,0,1,0) &= 6, & r_4(0,0,1,1) &= 10, \\ r_4(0,1,0,0) &= 3, & r_4(0,1,0,1) &= 15, & r_4(0,1,1,0) &= 5, & r_4(0,1,1,1) &= 9, \\ r_4(1,0,0,0) &= 1, & r_4(1,0,0,1) &= 13, & r_4(1,0,1,0) &= 7, & r_4(1,0,1,1) &= 11, \\ r_4(1,1,0,0) &= 2, & r_4(1,1,0,1) &= 14, & r_4(1,1,1,0) &= 4, & r_4(1,1,1,1) &= 8 \end{aligned}$$

This mapping r_n is one-to-one for every n : from the integer $r_n(x_1, x_2, \dots, x_n)$, we can recover its binary encoding (c_1, c_2, \dots, c_n) , from which we can recover first x_n , then x_{n-1} , and so on until x_1 .

To see that $r_n(\Phi_n(x)) = r_n(x) + 1 \bmod 2^n$, observe that (i) if x is the alternating vector $(\dots, 0, 1, 0, 1)$, then $r_n(x) = 2^n - 1$ and (ii) for all other vectors (x_1, x_2, \dots, x_n) in $\{0, 1\}^n$, the largest subscript m such that (x_1, x_2, \dots, x_m) is an alternating vector equals the smallest subscript m such that $c_m = 0$, in which case $r_n(x_1, x_2, \dots, x_n) + 1 = \sum_{j=1}^n d_j 2^{j-1}$ with

$$d_j = \begin{cases} 0 & \text{when } 1 \leq j < m, \\ 1 & \text{when } j = m, \\ c_j & \text{when } m < j \leq n \end{cases}$$

and, with (y_1, \dots, y_n) defined by (1), we have $r_n(y_1, \dots, y_n) = \sum_{j=1}^n d_j 2^{j-1}$.

4 How many n -neuron McCulloch-Pitts networks with period 2^n are there?

When $n = 2$, there are just two such networks,

$$(x_1, x_2) \mapsto (H(-x_2), H(x_1 - 1)) \quad \text{and} \quad (x_1, x_2) \mapsto (H(x_2 - 1), H(-x_1))$$

with H the Heaviside step function defined in Section 1. The first of these two networks is Φ_2 ; the second is produced from Φ_2 by switching the two coordinates.

Next, let us consider $n = 3$. By permuting the three coordinates, Φ_3

produces six distinct McCulloch-Pitts networks:

$$\begin{aligned}
&000 \rightarrow 100 \rightarrow 110 \rightarrow 010 \rightarrow 111 \rightarrow 011 \rightarrow 001 \rightarrow 101 \rightarrow 000, \\
&000 \rightarrow 010 \rightarrow 110 \rightarrow 100 \rightarrow 111 \rightarrow 101 \rightarrow 001 \rightarrow 011 \rightarrow 000, \\
&000 \rightarrow 001 \rightarrow 101 \rightarrow 100 \rightarrow 111 \rightarrow 110 \rightarrow 010 \rightarrow 011 \rightarrow 000, \\
&000 \rightarrow 001 \rightarrow 011 \rightarrow 010 \rightarrow 111 \rightarrow 110 \rightarrow 100 \rightarrow 101 \rightarrow 000, \\
&000 \rightarrow 010 \rightarrow 011 \rightarrow 001 \rightarrow 111 \rightarrow 101 \rightarrow 100 \rightarrow 110 \rightarrow 000, \\
&000 \rightarrow 100 \rightarrow 101 \rightarrow 001 \rightarrow 111 \rightarrow 011 \rightarrow 010 \rightarrow 110 \rightarrow 000.
\end{aligned}$$

(Here, we write $x_1x_2x_3$ for (x_1, x_2, x_3) and we appeal to the fact that an n -neuron McCulloch-Pitts network with period 2^n is fully specified by its trajectory.) By flipping bits ($0 \leftrightarrow 1$), these six McCulloch-Pitts networks produce an additional eighteen McCulloch-Pitts networks: flipping the first bit (and rotating the resulting trajectory to make 000 its starting point), we get

$$\begin{aligned}
&000 \rightarrow 010 \rightarrow 110 \rightarrow 011 \rightarrow 111 \rightarrow 101 \rightarrow 001 \rightarrow 100 \rightarrow 000, \\
&000 \rightarrow 011 \rightarrow 001 \rightarrow 101 \rightarrow 111 \rightarrow 100 \rightarrow 110 \rightarrow 010 \rightarrow 000, \\
&000 \rightarrow 011 \rightarrow 010 \rightarrow 110 \rightarrow 111 \rightarrow 100 \rightarrow 101 \rightarrow 001 \rightarrow 000, \\
&000 \rightarrow 001 \rightarrow 100 \rightarrow 101 \rightarrow 111 \rightarrow 110 \rightarrow 011 \rightarrow 010 \rightarrow 000, \\
&000 \rightarrow 010 \rightarrow 100 \rightarrow 110 \rightarrow 111 \rightarrow 101 \rightarrow 011 \rightarrow 001 \rightarrow 000, \\
&000 \rightarrow 001 \rightarrow 101 \rightarrow 011 \rightarrow 111 \rightarrow 110 \rightarrow 010 \rightarrow 100 \rightarrow 000;
\end{aligned}$$

flipping the second bit (and rotating the resulting trajectory to make 000 its starting point), we get

$$\begin{aligned}
&000 \rightarrow 101 \rightarrow 001 \rightarrow 011 \rightarrow 111 \rightarrow 010 \rightarrow 110 \rightarrow 100 \rightarrow 000, \\
&000 \rightarrow 100 \rightarrow 110 \rightarrow 101 \rightarrow 111 \rightarrow 011 \rightarrow 001 \rightarrow 010 \rightarrow 000, \\
&000 \rightarrow 001 \rightarrow 010 \rightarrow 011 \rightarrow 111 \rightarrow 110 \rightarrow 101 \rightarrow 100 \rightarrow 000, \\
&000 \rightarrow 101 \rightarrow 100 \rightarrow 110 \rightarrow 111 \rightarrow 010 \rightarrow 011 \rightarrow 001 \rightarrow 000, \\
&000 \rightarrow 001 \rightarrow 011 \rightarrow 101 \rightarrow 111 \rightarrow 110 \rightarrow 100 \rightarrow 010 \rightarrow 000, \\
&000 \rightarrow 100 \rightarrow 010 \rightarrow 110 \rightarrow 111 \rightarrow 011 \rightarrow 101 \rightarrow 001 \rightarrow 000;
\end{aligned}$$

flipping the third bit (and rotating the resulting trajectory to make 000 its

starting point), we get

$$\begin{aligned}
&000 \rightarrow 100 \rightarrow 001 \rightarrow 101 \rightarrow 111 \rightarrow 011 \rightarrow 110 \rightarrow 010 \rightarrow 000, \\
&000 \rightarrow 010 \rightarrow 001 \rightarrow 011 \rightarrow 111 \rightarrow 101 \rightarrow 110 \rightarrow 100 \rightarrow 000, \\
&000 \rightarrow 100 \rightarrow 101 \rightarrow 110 \rightarrow 111 \rightarrow 011 \rightarrow 010 \rightarrow 001 \rightarrow 000, \\
&000 \rightarrow 010 \rightarrow 011 \rightarrow 110 \rightarrow 111 \rightarrow 101 \rightarrow 100 \rightarrow 001 \rightarrow 000, \\
&000 \rightarrow 110 \rightarrow 100 \rightarrow 101 \rightarrow 111 \rightarrow 001 \rightarrow 011 \rightarrow 010 \rightarrow 000, \\
&000 \rightarrow 110 \rightarrow 010 \rightarrow 011 \rightarrow 111 \rightarrow 001 \rightarrow 101 \rightarrow 100 \rightarrow 000.
\end{aligned}$$

Since Φ_3 is antisymmetric, flipping two or three bits produces no additional McCulloch-Pitts networks. To summarize, Φ_3 produces an isomorphism class of 24 networks, where ‘isomorphism’ means any composition of permuting subscripts and flipping bits. The McCulloch-Pitts network defined by

$$(x_1, x_2, x_3) \mapsto (H(-x_1 + x_2 - x_3), H(-x_1 - x_2 - x_3 + 1), H(-x_1 + x_2 + x_3 - 1))$$

does not belong to this class: its trajectory is

$$000 \mapsto 110 \mapsto 100 \mapsto 010 \mapsto 111 \mapsto 001 \mapsto 011 \mapsto 101 \mapsto 000.$$

By permuting subscripts and flipping bits, this new network produces a new isomorphism class of 24 networks. Computer search shows that there are no 3-neuron McCulloch-Pitts networks with period 8 other than these 48 networks in these 2 isomorphism classes.

Additional computer search shows that there are precisely 9984 distinct 4-neuron McCulloch-Pitts networks with period 16 and that these networks come in 56 distinct isomorphism classes. It seems that the number of isomorphism classes of n -neuron McCulloch-Pitts networks with period 2^n grows rapidly with n .

5 Pseudorandom number generators

Our Φ_n has period 2^n , as long as could possibly be, but it is not quite irregular, disorderly, and apparently unpredictable. To point out two of its blatant blemishes, let y_i denote the i -th bit of a vector y in $\{0, 1\}^n$.

Our recursive description of the trajectory

$$(0, 0, \dots, 0) \mapsto \Phi_n(0, 0, \dots, 0) \mapsto \dots \Phi_n^{N-1}(0, 0, \dots, 0)$$

with $N = 2^n$ implies first that

$$(0, 0, \dots, 0) \neq \Phi_n(0, 0, \dots, 0), \quad \Phi_n^{N-2}(0, 0, \dots, 0) \neq \Phi_n^{N-1}(0, 0, \dots, 0)$$

and then that

$$x_1 = \Phi_n(x)_1 \Rightarrow \Phi_n(x)_1 \neq \Phi_n^2(x)_1. \quad (10)$$

Our recursive description of the trajectory also implies that for all $i = 1, 2, \dots, n$, the trajectory splits up into segments of length 2^{i-1} so that the i -th bit of $\Phi_n^t(0, 0, \dots, 0)$ is constant in each segment; it follows that

$$\text{for all } i = 2, 3, \dots, n, \quad x_i \neq \Phi_n(x)_i \Rightarrow \Phi_n(x)_i = \Phi_n^2(x)_i. \quad (11)$$

Properties (10), (11) make Φ_n far from irregular, disorderly, and apparently unpredictable: in a random permutation Φ of $\{0, 1\}^n$, we would expect $x_1 = \Phi(x)_1 = \Phi^2(x)_1$ about 25% of the time and, for each $i = 2, 3, \dots, n$, we would expect $x_i \neq \Phi(x)_i \neq \Phi^2(x)_i$ about 25% of the time.

By definition, no computable function $g : X \rightarrow X$ (where X is a finite set of numbers) can generate random numbers in the sequence $s, g(s), g^2(s), \dots$: if $g(x)$ can be computed, then it is not unpredictable. (For an exposition of the concept of randomness, we recommend [6].) Functions $g : X \rightarrow X$ that seem to generate random numbers are called *pseudorandom number generators*. Since each vector in $\{0, 1\}^n$ is a binary encoding of an n -bit nonnegative integer, every mapping $\Phi : \{0, 1\}^n \rightarrow \{0, 1\}^n$ induces a mapping $\Phi^* : \{0, 1, \dots, 2^n - 1\} \rightarrow \{0, 1, \dots, 2^n - 1\}$, and so every irregular, disorderly, apparently unpredictable McCulloch-Pitts network Φ induces a pseudorandom number generator Φ^* . Let us write

$$X_n = \{k/2^n : k = 0, 1, \dots, 2^n - 1\}.$$

Scaling down Φ^* by the factor of 2^n , we get a mapping $g_\Phi : X_n \rightarrow X_n$; for large values of n , this mapping approximates a pseudorandom number generator $g : [0, 1) \rightarrow [0, 1)$.

Long period alone does not suffice to make a pseudorandom number generator acceptable; in order to be acceptable, it has to pass a number of statistical tests. A number of these tests is commonly agreed on; our favourite ones are implemented in the software library TestU01 of L'Ecuyer and Simard[14, 15]. In particular, TestU01 includes batteries of statistical tests for sequences of uniform random numbers in the interval $[0, 1)$. The least stringent of them, **SmallCrush**, consists of the following ten tests:

```

1 smarsa_BirthdaySpacings
2 sknuth_Collision
3 sknuth_Gap
4 sknuth_SimpPoker
5 sknuth_CouponCollector
6 sknuth_MaxOft
7 svara_WeightDistrib
8 smarsa_MatrixRank
9 sstring_HammingIndep
10 swalk_RandomWalk1

```

Each of these tests produces a number p ; a typical range where the test is considered passed is $0.001 \leq p \leq 0.999$.

Is there a McCulloch-Pitts network Φ whose pseudorandom number generator g_Φ passes all ten tests of **SmallCrush**? Our networks Φ_n fail all ten.

We conclude with an example of a 4-neuron McCulloch-Pitts network which has neither of the two properties (10), (11). This network is defined by

$$(x_1, x_2, x_3, x_4) \mapsto (H(-x_1 - x_2 - 2x_3 - x_4 + 2), H(-x_1 - x_2 + x_3 + 2x_4 - 1), \\ H(-x_1 + 2x_2 - x_3 + x_4 - 1), H(2x_1 - x_2 - x_3 + x_4 - 1))$$

and its trajectory is

$$(0, 0, 0, 0) \mapsto (1, 0, 0, 0) \mapsto (1, 0, 0, 1) \mapsto (1, 1, 0, 1) \mapsto \\ (0, 0, 1, 1) \mapsto (0, 1, 0, 0) \mapsto (1, 0, 1, 0) \mapsto (0, 0, 0, 1) \mapsto \\ (1, 1, 1, 1) \mapsto (0, 1, 1, 1) \mapsto (0, 1, 1, 0) \mapsto (0, 0, 1, 0) \mapsto \\ (1, 1, 0, 0) \mapsto (1, 0, 1, 1) \mapsto (0, 1, 0, 1) \mapsto (1, 1, 1, 0) \mapsto \\ (0, 0, 0, 0) \mapsto (1, 0, 0, 0) \mapsto \dots$$

For each subscript $i = 1, 2, 3$, the triples $(x_i, \Phi(x)_i, \Phi^2(x)_i)$ run through the entire set $\{0, 1\}^3$; the triples $(x_4, \Phi(x)_4, \Phi^2(x)_4)$ miss only two values, which are $(0, 1, 0)$ and $(1, 0, 1)$.

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